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MIXED AND ISOPERIMETRIC ESTIMATES ON THE LOG-SOBOLEV CONSTANTS OF GRAPHS AND MARKOV CHAINS

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Two types of lower bounds are obtained on the log-Sobolev constants of graphs and Markov chains. The first is a mixture of spectral gap and logarithmic isoperimetric constant, the second involves the Gaussian isoperimetric constant. The sharpness of both types of bounds is tested on some examples. Product generalizations of some of these results are also briefly given.

1. Introduction

Log-Sobolev inequalities are playing a more important role in finite settings since they provide, for example, information on the mixing time of Markov chains which is more precise than spectral gap estimates (see Diaconis and Saloff–Coste [7], Saloff–Coste [18], Frieze and Kannan [8] and the references therein). More generally, large deviation results obtained via the log-Sobolev methodology can strengthen results available via "the bounded difference method". The log-Sobolev techniques can thus nicely complement eigenvalues techniques but have the drawback that the corresponding constants are reputedly hard to compute (dixum my combinatorics friends). Only in a few cases, starting with the work of Bonami [6] and Gross [9] on the discrete cube, are these constants exactly known (again see [18] for up to date information and further references). In contrast to the spectral gap which can be characterized (hence computed algorithmically or not) as the smallest

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non-zero eigenvalue of a matrix, such a methodology does not seem to be available in the log-Sobolev setting.

Besides this minimax characterization, there exist however other useful bounds on the spectral gap that can potentially be extended to the log-Sobolev setting. What we have in mind are Cheeger type inequalities involving isoperimetric constants (also known as conductance) see, for example, Alon [1], Alon and Milman [2], Jerrum and Sinclair [12], Sinclair and Jerrum [19], Kannan [13] and the references therein. Below, we provide logarithmic extensions of this type of isoperimetric bounds. To do so, we are aided by the work of Rothaus [17] as well as the more recent work of Ledoux [14] who introduced in a Riemannian geometric framework a Gaussian type isoperimetric quantity to lower bound the log-Sobolev constant. As we have already tried to convey in [10], [11] and [5]; in the discrete setting the key to this estimation question depends on the various notions of (lengths of) discrete gradient. A brief description of the present paper is as follows: In the next section, we introduce our framework and briefly recall some notions and results. In the third section we prove a discrete version of a theorem of Rothaus which leads to mixture type estimates on the log-Sobolev constant. In Section 4, we obtain a purely isoperimetric estimate. In the final section product generalizations are briefly discussed.

2. Preliminaries

Before presenting our results, let us recall our framework (throughout we only deal with finite space, but the infinite case works as well). X is a finite set equipped with a probability measure π and K is a non–negative kernel, $K\colon X\times X\longrightarrow \mathbf{R}^+$ (of course π could also be chosen to a finite positive measure). Throughout, and just for familiarity, K is either the transition matrix of a finite Markov chain (in which case K is Markovian, i.e., $\sum_{y\in X} K(x,y)=1$) or K is the adjacency matrix of an undirected graph $G=(V,E)$, i.e., for $x,y\in V$, $K(x,y)=1$ if $x\sim y$, and 0 otherwise. Associated with (X,K,π) is a family of discrete gradients say for $1\le p\le+\infty$, (the case $0<p<1$, could also be considered) which are defined for each $x\in X$ by

$$|\nabla_p f|(x)=\left(\sum_{y\in X}|f(x)-f(y)|^p K(x,y)\right)^{1/p},$$

$$\nabla_p^+ f(x)=\left(\sum_{y\in X}\left((f(x)-f(y))^+\right)^p K(x,y)\right)^{1/p},$$

and

$$\nabla_p^- f(x) = \left(\sum_{y \in X} ((f(x) - f(y))^-)^p \, K(x,y) \right)^{1/p},$$

for $p < +\infty$; and with the usual modification for $p = +\infty$, e.g.,

$$\nabla_\infty^+ f(x) = \sup_{y: K(x,y) > 0} (f(x) - f(y))^+.$$

Let us stop a moment and provide some intuition for these definitions, when $G = (V, E)$ and $K = \mathcal{A}$ the adjacency matrix of the graph G. Applying these definitions to $f = \mathbf{1}_A$ where $A \subset V$, we see that for $x \in A$, $\nabla_1^+ \mathbf{1}_A(x)$ is just the number of neighbors of x in A^c (equivalently the number of out-bound edges between x and A^c), while $\nabla_1^- \mathbf{1}_A(x)$ is the numbers of in-bound edges between $x \in A^c$ and A. For $p = 2$, we have the square root of these numbers. For $p = +\infty$ and $x \in A$, $\nabla_\infty^+ \mathbf{1}_A(x)$ is one or zero whether or not there is (at least) a vertex in A^c linked to $x \in A$ and similarly for $\nabla_\infty^- \mathbf{1}_A(x)$. Now, taking expectation with respect to, say, the uniform measure π on V, we see that $E \nabla_1^+ \mathbf{1}_A$ is the normalized number of out-bound edges between A and A^c, while $E \nabla_\infty^+ \mathbf{1}_A$ is the normalized number of vertices in A which have at least one neighbor in A^c. Similar results hold for the other cases. Let us just mention that the $|\nabla.|$ are symmetric ways of counting edges or vertices and so for a fixed p, ∇_p^+ or ∇_p^- can be significantly smaller that $|\nabla_p|$. This spectrum of gradients allows us to deal with problems of edge to vertex isoperimetry by considering different p.

The properties of the kernel determine the growth of the discrete gradients, e.g., when K is Markovian; $|\nabla_1| \le |\nabla_2| \le |\nabla_\infty|$, while when $K = \mathcal{A}$, $|\nabla_\infty| \le |\nabla_2| \le |\nabla_1|$; and similarly with the $+$ and $-$ versions (note however that in the infinity versions, the kernel is important only as long as it is zero or not). It thus follows that the various functional constants we consider do satisfy similar inequalities. We should also point out that the extremal cases $p = 1$ and $+\infty$, play a role similar to one another in their respective framework Markovian ($p = 1$); graph ($p = +\infty$). Since all the results presented here are symmetric in that they are true for both extremal cases, this recovers the Markov and Graph case at once.

Recall that for any $f : X \longrightarrow \mathbf{R}$; $\mathrm{Ent} f^2 = E f^2 \log f^2 - E f^2 \log E f^2$ denotes the entropy of f^2 (throughout, E and Var are always taken with respect to π and $0 \log 0 = 0$). (X, K, π) is said to satisfy a log-Sobolev inequality with constant $\rho > 0$ if

$$\rho \, \mathrm{Ent} f^2 \le E(\mathrm{D} f)^2, \tag{2.1}$$

for all $f: X \longrightarrow \mathbf{R}$ and where $\mathrm{D}f$ denotes any of the discrete gradient defined above. Replacing $\mathrm{Ent} f^2$ by $\mathrm{Var} f$ in the above inequality we correspondingly get a Poincaré inequality with constant λ; and clearly $\lambda \geq 2\rho$, since $\lim_{C\to+\infty} \mathrm{Ent}(f+C)^2 = 2\ \mathrm{Var} f$. When needed, to emphasize the dependency on the particular discrete gradient used, we specifically write $\rho_p, \rho_p^+, \ldots$. For example, ρ_2 is twice the usual log-Sobolev constant associated to the Dirichlet form of a symmetric Markov chain (X, K, π). To finish this preliminary section, we present co-area formulas (proofs of these results as well as a more detailed analysis of the various log-Sobolev and Poincaré constants are in [10], [11], [5]).

Lemma 1. *For any* $f: X \longrightarrow \mathbf{R}$,

$$ E\mathrm{D}f = \int_{-\infty}^{+\infty} E\mathrm{D}\mathbf{1}_{f>t}dt, \tag{2.2}$$

where D *denotes either one of* $\nabla_1^{\pm}$ *or* $\nabla_{\infty}^{\pm}$.

Remark 1. Outside of the cases $\nabla_1^{\pm}$ and $\nabla_{\infty}^{\pm}$, note that (2.2) continues to hold for $|\nabla_1| = \nabla_1^+ + \nabla_1^-$, as previously shown by Tillich [20]. Moreover, for $|\nabla_\infty| = \max(\nabla_\infty^+, \nabla_\infty^-)$, the above equality becomes

$$ E|\nabla_\infty f| \leq E(\nabla_\infty^+ f + \nabla_\infty^- f) = \int_{-\infty}^{+\infty} E|\nabla_\infty \mathbf{1}_{f>t}|dt \leq 2E|\nabla_\infty f|, $$

where the equality follows from (2.2) and since for indicator functions, ∇_∞^+ and ∇_∞^- have disjoint supports.

Unfortunately (as shown below) outside of the cases $p=1$ and $p=+\infty$, such co–area (in)equalities do not, in general, hold.

Let us now present an example showing that a co-area type inequality

$$ E\nabla_2^+ f \geq C \int_{-\infty}^{+\infty} E\nabla_2^+ \mathbf{1}_{f>t}dt, \tag{2.3}$$

for all $f: X \longrightarrow \mathbf{R}$ and some absolute constant $C > 0$, does not necessarily hold. For $X = \{x_1, \ldots, x_n\}$, K and π to be chosen later, let us rewrite both terms in (2.3). Assuming that $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n)$, the left hand side of (2.3) is

$$ E\nabla_2^+ f = \sum_{i=1}^n \sqrt{\sum_{j=1}^n (f(x_i) - f(x_j))^{+2} K(x_i, x_j)\pi(x_i)} $$

$$= \sum_{i=2}^{n} \sqrt{\sum_{j=1}^{i-1} (f(x_i) - f(x_j))^2 K(x_i, x_j)\pi(x_i)}. \tag{2.4}$$

Now for the right hand side of (2.3), we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} E\nabla_2^+ \mathbf{1}_{f>t} dt \\
&= \int_{-\infty}^{f(x_1)} + \int_{f(x_n)}^{+\infty} \left(\sum_{i=1}^{n} \sqrt{\sum_{j=1}^{n} \left(\mathbf{1}_{f>t}(x_i) - \mathbf{1}_{f>t}(x_j)\right)^+ K(x_i, x_j)\pi(x_i)} \right) dt \\
&\quad + \sum_{k=1}^{n-1} \int_{f(x_k)}^{f(x_{k+1})} \sum_{i=1}^{n} \sqrt{\sum_{j=1}^{n} \left(\mathbf{1}_{f>t}(x_i) - \mathbf{1}_{f>t}(x_j)\right)^+ K(x_i, x_j)\pi(x_i) dt}
\end{aligned}$$

$$= \sum_{k=1}^{n-1} (f(x_{k+1}) - f(x_k)) \sum_{i=k+1}^{n} \sqrt{\sum_{j=1}^{k} K(x_i, x_j)\pi(x_i)}. \tag{2.5}$$

Now let X be the vertex set of the complete graph on n vertices, let $K(x,y) = \frac{1}{n-1}$ for $x \neq y$ and let $\pi(x) = \frac{1}{n}$; for all $x \in X$. For this (X, K, π), (2.4) and (2.5) respectively become

$$\frac{1}{n\sqrt{n-1}} \sum_{k=1}^{n-1} \sqrt{\sum_{j=1}^{k} (f(x_{k+1}) - f(x_j))^2}, \tag{2.6}$$

and

$$\frac{1}{n\sqrt{n-1}} \sum_{k=1}^{n-1} (f(x_{k+1}) - f(x_k))(n-k)\sqrt{k}. \tag{2.7}$$

So proving a co-area type inequality (2.3) is equivalent to showing that

$$C \sum_{k=1}^{n-1} (f(x_{k+1}) - f(x_k))(n-k)\sqrt{k} \leq \sum_{k=1}^{n-1} \sqrt{\sum_{j=1}^{k} (f(x_{k+1}) - f(x_j))^2}, \tag{2.8}$$

for some positive $C < +\infty$ and all f such that $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n)$.

However this is clearly false. Indeed, taking $n-1 = 13^\ell$ and

$$f(x_{k+1}) - f(x_k) = \begin{cases} \frac{1}{\sqrt{k}} & \text{if } k = 13^p \\ 0 & \text{otherwise} \end{cases}$$

we see that the left hand side of (2.8) is of order $\ell 13^\ell$ while the right hand side is dominated by

$$
\begin{aligned}
13^\ell \sqrt{\sum_{k=1}^{13^\ell} (f(x_{13^\ell+1}) - f(x_k))^2} &\approx 13^\ell \sqrt{(1-q)^2 + (1-q^2)^2 + \cdots + (1-q^\ell)^2} \\
&\le 13^\ell \sqrt{\ell(1-q^\ell)^2} \\
&= 13^\ell \sqrt{\ell}(1-q^\ell),
\end{aligned}
$$

where $q = \dfrac{1}{\sqrt{13}}$. ∎

3. Mixed bounds

Following Rothaus [17] (see also [4], [20]), let G be a non-empty set of pairs (g_1, g_2) of functions on X and let $\mathcal{L}$ be a functional generated by G via

$$
\mathcal{L}(f) = \sup_{(g_1,g_2)\in G} E(f^+ g_1 + f^- g_2), \tag{3.1}
$$

where as usual $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Taking $g_1 = -g_2$ in (3.1), gives

$$
\mathcal{L}(f) = \sup_{g\in G} Efg. \tag{3.2}
$$

As noted in [17], many functionals have the representation (3.1) or (3.2). For example, $\mathcal{L}(f) = (E|f - Ef|^p)^{1/p}, 1 \le p \le +\infty$; $\mathcal{L}(f) = \inf_{a\in\mathbb{R}} E|f - a|$, i.e., $\mathcal{L}(f) = E|f - m(f)|$, where $m(f)$ is a median of f. Of particular interest to us is the entropy functional

$$
\mathrm{Ent}|f| = E|f|\log|f| - E|f|\log E|f|, \tag{3.3}
$$

which admits the representation (3.1) since $\mathrm{Ent}|f| = \sup_{Ee^g \le 1} E|f|g$. With these definitions and the co-area formulas we now have the discrete version of Rothaus' theorem.

Theorem 1. *Let $c \ge 0$. The following are equivalent:*
(i) $c\mathcal{L}(f) \le E\mathrm{D}f$, for all $f : X \longrightarrow \mathbf{R}$,
(ii) $c\mathcal{L}(\mathbf{1}_A) \le E\mathrm{D}\mathbf{1}_A$ and $c\mathcal{L}(-\mathbf{1}_A) \le E\mathrm{D}(-\mathbf{1}_A)$, for all $A \subset X$,
where D *is as in Lemma 1.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial, apply (i) to $f=\mathbf{1}_A$ and $f=-\mathbf{1}_A$ respectively. For the converse, let $(g_1,g_2)\in G$. Then, by the previous lemma and since $E\mathrm{D}(-\mathbf{1}_A)=E\mathrm{D}\mathbf{1}_{A^c}$,

$$\begin{aligned} E\mathrm{D}f &= \int_0^\infty E\mathrm{D}\mathbf{1}_{f>t}dt + \int_{-\infty}^0 E\mathrm{D}\mathbf{1}_{f>t}dt \\ &\geq c\int_0^\infty E\mathbf{1}_{f>t}g_1dt + \int_{-\infty}^0 E\mathrm{D}(-\mathbf{1}_{f\leq t})dt \\ &\geq cEf^+g_1 + c\int_{-\infty}^0 E\mathbf{1}_{f\leq t}g_2dt \\ &= cE(f^+g_1+f^-g_2), \end{aligned}$$

from which the result follows. ∎

Remark 2. Some special cases of the above result are worth mentioning:
(i) If $\mathcal{L}$ is even, i.e., $\mathcal{L}(f)=\mathcal{L}(-f)$, then the condition (ii) becomes

$$c\mathcal{L}(\mathbf{1}_A) \leq \min(E\mathrm{D}\mathbf{1}_A, E\mathrm{D}(-\mathbf{1}_A)) = \min(E\mathrm{D}\mathbf{1}_A, E\mathrm{D}\mathbf{1}_{A^c}).$$

(ii) If $E\mathrm{D}\mathbf{1}_A=E\mathrm{D}(-\mathbf{1}_A)$ (this happens in particular if the Markov chain is reversible or in the case of undirected graphs if π is uniform) then (ii) reads

$$c\max(\mathcal{L}(\mathbf{1}_A),\mathcal{L}(-\mathbf{1}_A)) \leq E\mathrm{D}\mathbf{1}_A,$$

for all $A\subset X$.
(iii) If D is replaced by $|\nabla_1|$, then the theorem remains valid, a result previously obtained by Tillich [20] not only for functionals $\mathcal{L}$ as above but also for semi–norms (for which Theorem 1 is also true). Replacing D by $|\nabla_\infty|$ gives a similar result up to a multiplicative constant.

The main point to draw from Theorem 1 is that in order to verify a functional inequality

$$c\mathcal{L}(f) \leq E\mathrm{D}f,$$

for all $f: X\longrightarrow \mathbf{R}$, it is enough to verify it on $f=\mathbf{1}_A$ and $f=-\mathbf{1}_A$ and that the optimal constant c can be correspondingly found. For example, if $\mathcal{L}(f)=\mathrm{Ent}f=Ef\log f - Ef\log Ef$, $f\geq 0$, then the optimal constant c in $c\mathrm{Ent}(f-m(f))^+\leq E\mathrm{D}f$, for all $f:X\longrightarrow\mathbf{R}$, (equivalently for all $f\geq 0$), and where $m(f)$ is a median of f for π; is the *logarithmic isoperimetric constant* (or the *logarithmic conductance* to use a more discrete terminology)

$$\inf_{0<\pi(A)\leq 1/2} \frac{E\mathrm{D}\mathbf{1}_A}{-\pi(A)\log\pi(A)}.$$

The previous equivalence leads us to the main result of this section. Before proving it, let us set a bit more precisely the notations. For any $1\leq p\leq +\infty$, let

$$\ell_p^+ = \inf_{0<\pi(A)\leq 1/2} \frac{E\nabla_p^+\mathbf{1}_A}{-\pi(A)\log\pi(A)}; \tag{3.4}$$

with similar definitions for ℓ_p^-, replacing ∇_p^+ by ∇_p^-. From now on it is assumed that the constants (spectral gap, logarithmic conductance, ...) are positive, this usually holds under an irreducibility assumption.

Theorem 2.

$$\rho_p \geq \frac{\lambda_p \min(\ell_1^+,\ell_\infty^+)}{2\left(\sqrt{2\lambda_p} + 2\min(\ell_1^+,\ell_\infty^+)\right)}, \qquad 1<p<+\infty. \tag{3.5}$$

$$\rho_p \geq \frac{\lambda_p \ell_p^+}{2\left(\sqrt{2\lambda_p} + 2\ell_p^+\right)}, \qquad p=1,+\infty. \tag{3.6}$$

$$\rho_p \geq \frac{\lambda_p \min(\ell_1^-,\ell_\infty^-)}{2\left(\sqrt{2\lambda_p} + 2\min(\ell_1^-,\ell_\infty^-) + \lambda_p\right)}, \tag{3.7}$$

$2\leq p<+\infty$ in the Markov case, and $1<p<+\infty$ in the graph case.

$$\rho_p \geq \frac{\lambda_p \ell_p^-}{2\left(\sqrt{2\lambda_p} + 2\ell_p^- + \lambda_p\right)}, \tag{3.8}$$

$p=+\infty$, or $p=1$ and in the graph case.

Proof. We first prove (3.5) in the Markovian case (where, $\min(\ell_1^+,\ell_\infty^+)=\ell_1^+$) and then (3.7) in the graph case (where, $\min(\ell_1^+,\ell_\infty^+)=\ell_\infty^+$). The proofs of the other cases and of (3.6) and (3.8) are similar and so omitted. Let us note nevertheless that the left hand sides in the above inequalities are non–decreasing in ℓ and so for $p=+\infty$ in the Markov case and $p=1$ in the graph case, (3.6) is better than (3.5) and (3.8) is better than (3.7).

Using the definition of ℓ_1^+ and Theorem 1, we see that ℓ_1^+ is the optimal constant c in the inequality $c\mathrm{Ent}(f-m(f))^+\leq E\nabla_1^+ f$, for all non–negative

f defined on X. Now, let f be such that $m(f)=0$ (thus $m(f^+)=m(f^-)=m(f^{+2})=m(f^{-2})=0$). The very definition of ∇_1^+ gives

$$\ell_1^+ \mathrm{Ent} f^{+2} \leq E\nabla_1^+ f^{+2} \leq 2Ef^+\nabla_1^+ f^+,$$

and

$$\ell_1^+ \mathrm{Ent} f^{-2} \leq E\nabla_1^+ f^{-2} \leq 2Ef^-\nabla_1^+ f^-,$$

Next, it is easy to verify that $\nabla_1^+ f^+ \leq |\nabla_1 f|\mathbf{1}_{f>0}$, and that $\nabla_1^+ f^- \leq |\nabla_1 f|\mathbf{1}_{f<0}$. Hence,

$$\ell_1^+ \left(\mathrm{Ent} f^{+2} + \mathrm{Ent} f^{-2}\right) \leq 2E|f||\nabla_1 f|.$$

But, $\mathrm{Ent} f^2 \leq \mathrm{Ent} f^{+2} + \mathrm{Ent} f^{-2}$, thus by the Cauchy–Schwarz inequality

$$\ell_1^+ \mathrm{Ent} f^2 \leq 2\sqrt{Ef^2}\sqrt{E|\nabla_1 f|^2}.$$

Now, $E(f-m(f))^2 = \mathrm{Var} f + (E(f-m(f)))^2 \leq 2\mathrm{Var} f$, thus for any f, we have

$$\ell_1^+ \mathrm{Ent}(f-m(f))^2 \leq 2\sqrt{2}\sqrt{\mathrm{Var} f}\sqrt{E|\nabla_1 f|^2}.$$

Since K is Markovian this lead to,

$$\ell_1^+ \mathrm{Ent}(f-m(f))^2 \leq \frac{2\sqrt{2}}{\sqrt{\lambda_p}} E|\nabla_p f|^2. \tag{3.9}$$

By an inequality of Rothaus [17] (Lemma 9) $\mathrm{Ent} f^2 \leq \mathrm{Ent}(f-m(f))^2 + 2E(f-m(f))^2$, which when combined with the previous arguments, in particular (3.9), gives:

$$\ell_1^+ \mathrm{Ent} f^2 \leq \left(\frac{2\sqrt{2}}{\sqrt{\lambda_p}} + \frac{4\ell_1^+}{\lambda_p}\right) E|\nabla_p f|^2,$$

from which (3.5) follows.

Let us now prove (3.7) in the graph case (where $\nabla_\infty^+ \leq \nabla_p^+, 1 \leq p$, and so $\min(\ell_1^+, \ell_\infty^+) = \ell_\infty^+$). Again, for any f such that $m(f)=0$, $\ell_\infty^- \mathrm{Ent} f^{+2} \leq E\nabla_\infty^- f^{+2}$, and it follows by the very definition of ∇_∞^- that,

$$\ell_\infty^- \mathrm{Ent} f^{+2} \leq E(\nabla_\infty^- f^+)^2 + 2Ef^+\nabla_\infty^- f^+,$$

and similarly,

$$\ell_\infty^- \mathrm{Ent} f^{-2} \leq E(\nabla_\infty^- f^-)^2 + 2Ef^-\nabla_\infty^- f^-.$$

But, $\nabla_\infty^- f^+ \le |\nabla_\infty f|$, $\nabla_\infty^- f^- \le |\nabla_\infty f|$, and $\mathrm{Ent} f^2 \le \mathrm{Ent} f^{+2} + \mathrm{Ent} f^{-2}$. Hence,

$$\begin{aligned}\ell_\infty^- \mathrm{Ent} f^2 &\le 2E|\nabla_\infty f|^2 + 2E|f||\nabla_\infty f|\\ &\le 2E|\nabla_\infty f|^2 + 2\sqrt{Ef^2}\sqrt{E|\nabla_\infty f|^2}\\ &\le 2E|\nabla_p f|^2 + 2\sqrt{Ef^2}\sqrt{E|\nabla_p f|^2},\end{aligned}$$

where we also used the fact that we are in the graph case. Thus,

$$\begin{aligned}\ell_\infty^- \mathrm{Ent}(f-m(f))^2 &\le 2E|\nabla_p f|^2 + 2\sqrt{E(f-m(f))^2}\sqrt{E|\nabla_p f|^2}\\ &\le \left(2+\frac{2\sqrt{2}}{\sqrt{\lambda_p}}\right)E|\nabla_p f|^2, \qquad (3.10)\end{aligned}$$

for all f. Now proceeding as in the proof of (3.5), (3.10) leads to

$$\ell_\infty^- \mathrm{Ent} f^2 \le \left(2+\frac{2\sqrt{2}}{\sqrt{\lambda_p}}+\frac{4\ell_\infty^-}{\lambda_p}\right)E|\nabla_p f|^2,$$

from which (3.7) follows. ∎

Remark 3. (i) In the lower bounds, the absolute constants might not be optimal. However as shown below, the order in λ and ℓ is sharp. Note also that for arbitrary K the results continue to hold with worse constants (depending on K). Simple upper estimates of ρ can also be found by taking $f = \mathbf{1}_A$ in the defining property of ρ and using $\rho \le \frac{\lambda}{2}$. For example, for $p = +\infty$, we have

$$\rho_\infty \le \min\left(\frac{\lambda_\infty}{2}, \ell_\infty\right).$$

(ii)There is an unfortunate discrepancy, in that one would have liked to see the presence of $\ell_p^\pm$ in the left hand sides of (3.5)–(3.6). The lack of co–area (in)equality for $1 < p < +\infty$, partly explains this fact; we do not know if a functional equivalence theorem similar to Theorem 1 holds for this range of p.
(iii) In the above, and since the left hand sides are non–decreasing in λ, purely isoperimetric types of bounds can be obtained on the ρ's by replacing the λ's by the isoperimetric constants present in Cheeger's inequality (see [10] for various versions of Cheeger's inequality with λ_p). Note also that Theorem 2 provides another way of showing that for irreducible chains (or connected graphs) the log-Sobolev constant is positive.

(iv) Another type of refinement involving ℓ_1 and K Markovian is actually possible. Indeed, let

$$\ell_p = \inf_{0<\pi(A)\leq 1/2} \frac{E|\nabla_p \mathbf{1}_A|}{-\pi(A)\log\pi(A)}. \tag{3.11}$$

Then, note that since $-x\log x \geq -(1-x)\log(1-x), 0\leq x\leq 1/2$, and since $E|\nabla_p \mathbf{1}_A| = E|\nabla_p \mathbf{1}_{A^c}|$, actually $\ell_p = \inf_{0<\pi(A)<1} \frac{E|\nabla_p \mathbf{1}_A|}{-\pi(A)\log\pi(A)}$. Thus from Theorem 1 and since $|\nabla_1| = \nabla_1^+ + \nabla_1^-$, we see that ℓ_1 is the optimal constant c in $c\mathrm{Ent}|f| \leq E|\nabla_1 f|$, for all $f: X \longrightarrow \mathbf{R}$ or equivalently (since $|\nabla_1|f|| \leq |\nabla_1 f|$) in $c\mathrm{Ent} f \leq E|\nabla_1 f|$, for all $f \geq 0$. Let us now use this fact in a way similar to the proof of the previous results. Indeed, for any $f\geq 0$,

$$\begin{aligned} \ell_1 \mathrm{Ent} f^2 &\leq E|\nabla_1 f^2| \\ &\leq E|\nabla_2 f|^2 + 2Ef|\nabla_1 f|. \end{aligned}$$

Hence,

$$\begin{aligned} \ell_1 \mathrm{Ent}(f-Ef)^2 &\leq E|\nabla_2|f-Ef||^2 + 2E|f-Ef||\nabla_1|f-Ef|| \\ &\leq E|\nabla_2 f|^2 + 2\sqrt{E(f-Ef)^2}\sqrt{E|\nabla_1 f|^2} \\ &\leq \left(1+\frac{2}{\sqrt{\lambda_p}}\right) E|\nabla_p f|^2, \end{aligned}$$

for $2\leq p<+\infty$, since K is Markovian. Rothaus' inequality (in the form $\mathrm{Ent} f^2 \leq \mathrm{Ent}(f-Ef)^2) + 2\mathrm{Var} f)$ leads to

$$\rho_p \geq \frac{\lambda_p \ell_1}{2\left(\sqrt{\lambda_p}+\ell_1\right)+\lambda_p}, \tag{3.12}$$

for any $2\leq p<+\infty$. When (X,K,π) is reversible, $\ell_1 = 2\ell_1^+ = 2\ell_1^-$ and so (3.12) slightly improves (3.7). Actually, for reversible chains, and $f\geq 0$, $E|\nabla_1 f^2| \leq 2Ef|\nabla_1 f|$ and so proceeding as above gives:

$$\rho_p \geq \frac{\lambda_p \ell_1}{2\left(\sqrt{\lambda_p}+\ell_1\right)}, \tag{3.13}$$

$1\leq p<+\infty$. In an abstract framework, metric space with modulus of the gradient, a co–area inequality holds and so does the abstract version of Theorem 1 ([4]). Thus, since the chain rule holds, the proof of (3.13) carries over with ℓ_1 replaced by the corresponding quantity for the modulus of the gradient. For example for compact Riemannian manifolds with finite (normalized) Riemannian measure μ, this gives

$$\rho \geq \frac{\lambda \ell}{2\left(\sqrt{\lambda}+\ell\right)}, \tag{3.14}$$

where ρ is the usual log-Sobolev constant, λ the first eigenvalue of the Laplace operator and $\ell = \inf_{0<\mu(A)\leq 1/2} \frac{\mu(\partial A)}{-\mu(A)\log\mu(A)}$. As given, (3.14) complements recent results of Wang [21] and Ledoux [16] providing lower bound on ρ in terms of λ and of the diameter of a manifold. Moreover, in very recent work, Bobkov [3] obtained for compactly supported log-concave measures μ, a result related to (3.14). His Corollary 2.2. implies that as defined above ℓ is such that $\ell \geq 1/r$, where r is the radius of the Euclidean ball supporting μ. Let us also mention that other types of bounds are also available (see, for example, [14], [16], [17], [18], ...).

To finish these remarks, we note that coming back to graphs in which case ℓ_∞ is the smallest quantity, in similarity with (3.12) we have for $1<p\leq+\infty$,

$$\rho_p \geq \frac{\lambda_p \ell_\infty}{2\left(2\sqrt{\lambda_p}+\ell_\infty+\lambda_p\right)}, \tag{3.15}$$

where, as before, worse absolute constants appear because of the form of the co–area inequality in case $p=+\infty$. Similarly, for $p=1$ and still for graphs, we have

$$\rho_1 \geq \frac{\lambda_1 \ell_1}{\left(2\sqrt{\lambda_1}+\ell_1+\lambda_1\right)}. \tag{3.16}$$

Since the left hand sides of (3.12)–(3.16) are non decreasing in λ and since $x\log 2 \leq -x\log x, 0\leq x\leq 1/2$, Cheeger's inequality leads to the generic

$$\rho \geq C\ell^2, \tag{3.17}$$

for some absolute constant C. Actually this purely isoperimetric right hand side of (3.17) can be improved as done next and also in the next section. Before doing so, let us present various examples showing that the estimates obtained above can be sharp.

Below, G is a finite graph and π is the normalized counting measure on its vertex set V.

- Let $G = Q_n$ be the n-dimensional discrete cube. It is well known that $\lambda_2(Q_n)=2\rho_2(Q_n)=4$. Hence, $\rho_\infty(Q_1)=2$, and it is easy to check (see Section 5) that $\rho_\infty(Q_n) \geq \frac{2}{n}$. This shows that the general bound $\rho_\infty \geq \rho_2/\Delta(G)$, where $\Delta(G)$ is the maximal degree of the graph, is tight. This also shows that the estimates of the previous theorem are all, in general, tight (up to absolute constants). Indeed, $\ell_\infty^+(Q_n) \geq \frac{\ell_\infty^+(Q_1)}{\sqrt{n}} = \frac{1}{\log 2\sqrt{n}}$, $\ell_\infty^-(Q_n) \geq \frac{\ell_\infty^-(Q_1)}{\sqrt{n}} = \frac{1}{\log 2\sqrt{n}}$, and $\ell_\infty(Q_n) = 2\ell_\infty^\pm(Q_n)$ and similarly $\ell_1^\pm(Q_n) = \ell_1^\pm(Q_1) = \frac{1}{\log 2}$ and $\ell_1(Q_n) = 2\ell_1^\pm(Q_n)$.

• Let $G = K_n$ be the complete graph on n vertices; and let n be even for convenience. Then $\ell_\infty^+(K_n) = \frac{1}{\log n}$, $\ell_\infty^-(K_n) = \frac{1}{\log 2}$ and $\ell_\infty(K_n) = \frac{2}{\log 2}$. This shows that the general inequality $\ell_\infty \geq \ell_\infty^+ + \ell_\infty^-$ can be tight. Actually, it is easily verified that $\lambda_\infty(K_n)$ and $\rho_\infty(K_n)$ are bounded above and below independently of n. Thus, in contrast to (3.6), (3.8) is sharp. Also, $\lambda_2 = 2n$ and thus, $\rho_2(K_n) \geq \frac{2n}{4(\sqrt{2n}\log 2 + 2)}$, while $\rho_\infty(K_n) \leq \frac{2}{\log 2}$, showing that ρ_2 can become unbounded, while ρ_∞ cannot, as shown next. Indeed, taking in the defining property of $\rho_\infty(G)$, $f = \mathbf{1}_A$, gives $\rho_\infty(G) \leq \inf_{0<\pi(A)\leq 1/2} \frac{1}{-\pi(A)\log \pi(A)}$. In particular, if for some $A \subset V$, $\pi(A) = 1/2$, then $\rho_\infty(G) \leq \frac{2}{\log 2}$. This is the case when π is the normalized counting measure and $|V| = 2n$ is even. If $|V| = 2n+1$, $n = 1, 2, 3 \ldots$, then taking $\pi(A) = n/(2n+1)$ gives

$$\rho_\infty(G) \leq \frac{(2n+1)}{n \log\left(\frac{2n+1}{n}\right)}. \tag{3.18}$$

The sequence on the right in (3.18) is increasing in $n \geq 2$, tends to $2/\log 2$ and for $n=1$ is equal to $3/\log 3$. Thus, for all n,

$$\rho_\infty(G) \leq \frac{2}{\log 2}. \tag{3.19}$$

Therefore, this estimate holds for all finite graphs with the normalized counting measure.

Returning to a Markov chain framework (the simple random walk on K_n), It is known (see [7]) that $\rho_2(K_n) = \frac{2(n-2)}{(n-1)\log(n-1)}$. Since as easily computed ℓ_1^+ is of order $1/\log n$, we see that (3.6) with $p=1$ can be of the right order.

Theorem 2 can be improved. To do so, for any fixed $\varepsilon \geq 0$, let

$$\ell_p^+(\varepsilon) = \inf_{0<\pi(A)\leq\frac{1}{2}} \left\{ \frac{E\nabla_p^+ \mathbf{1}_A}{-\pi(A)\log\pi(A)} + \frac{\varepsilon}{-\log\pi(A)} \right\}, \tag{3.20}$$

$1 \leq p \leq +\infty$, with similar definitions for $\ell_p^-(\varepsilon)$ and $\ell_p(\varepsilon)$. Clearly, $\ell_p^+(0) = \ell_p^+$ and

$$\ell_p^+(\varepsilon) \geq \ell_p^+ + \varepsilon \inf_{0<\pi(A)\leq\frac{1}{2}} \left\{ \frac{1}{-\log\pi(A)} \right\}. \tag{3.21}$$

where in general this last type of inequality is strict (for the complete graph, $\ell_\infty^-(\varepsilon) = \dfrac{1+\varepsilon}{\log 2}$, for $\varepsilon \leq 2\log 2 - 1$ and $\dfrac{n-1+\varepsilon}{\log n}$, for $\varepsilon \geq n\log n - (n-1)$, while the corresponding version of the right hand side of (3.21) equals $\dfrac{1}{\log 2} + \dfrac{\varepsilon}{\log n}$).

Now from the method of Theorem 1, (or simply using Lemma 1 in conjunction with $Ef = \int_0^{+\infty} E\mathbf{1}_{f>t}dt$ and $\operatorname{Ent} f \leq -\int_0^{+\infty} E\mathbf{1}_{f>t}\log E\mathbf{1}_{f>t}dt$, which follows from elementary measure theory for any $f \geq 0$), it is clear that $\ell_1^+(\varepsilon)$ is the optimal constant $c(\varepsilon)$ in

$$c(\varepsilon)\ \operatorname{Ent}(f - m(f))^+ \leq E(\nabla_1^+ f) + \varepsilon E(f - m(f))^+ \tag{3.22}$$

for all f (or $f \geq 0$). In other words,

$$\ell_p^+(\varepsilon) = \inf_{f \neq \text{constant}} \left\{ \frac{E\nabla_p^+ f}{\operatorname{Ent}(f - m(f))^+} + \varepsilon \frac{E(f - m(f))^+}{\operatorname{Ent}(f - m(f))^+} \right\}.$$

Assuming a Markovian framework and proceeding as in the proof of Theorem 2 applying (3.22) to f^{+2}, f^{-2} where f is such that $m(f) = 0$, we get instead of (3.9)

$$\ell_1^+(\varepsilon)\operatorname{Ent}(f - m(f))^2 \leq \frac{2\sqrt{2}}{\sqrt{\lambda_p}} E|\nabla_p f|^2 + \varepsilon E(f - m(f))^2, \tag{3.23}$$

for all f. In turn, (3.23) gives

$$\ell_1^+(\varepsilon)\operatorname{Ent}\ f^2 \leq \left(\frac{2\sqrt{2}}{\sqrt{\lambda_p}} + \frac{2\varepsilon}{\lambda_p} + \frac{4\ell_1^+(\varepsilon)}{\lambda_p} \right) E|\nabla_p f|^2$$

Hence,

$$\rho_p \geq \sup_{\varepsilon \geq 0} \frac{\ell_1^+(\varepsilon)\lambda_p}{2\left(\sqrt{2\lambda_p} + \varepsilon + 2\ell_1^+(\varepsilon)\right)}. \tag{3.24}$$

This leads to the following theorem (whose proof is clear from the above arguments) and which is stated only for $p=1$ and $+\infty$ (the other cases follow by replacing below $\ell_p^+(\varepsilon)$ by $\min(\ell_p^+(\varepsilon), \ell_\infty^+(\varepsilon))$ and similarly for $\ell_p^-(\varepsilon)$). Of course, ε-versions of (3.12)–(3.16) also hold.

Theorem 3.

$$\rho_p \geq \sup_{\varepsilon\geq 0} \frac{\lambda_p \ell_p^+(\varepsilon)}{2\left(\sqrt{2\lambda_p}+\varepsilon+2\ell_p^+(\varepsilon)\right)}, \qquad p=1,+\infty \tag{3.25}$$

$$\rho_p \geq \sup_{\varepsilon\geq 0} \frac{\lambda_p \ell_p^-(\varepsilon)}{2\left(\sqrt{2\lambda_p}+\varepsilon+2\ell_\infty^-(\varepsilon)+\lambda_p\right)}, \tag{3.26}$$

$p=+\infty$, or $p=1$ and in the graph case.

Remark 4. (i) Theorem 3 is not just a cosmetic improvement of Theorem 2. Indeed let $p=+\infty$, let $X=\{0,1\}$ with $\pi(1)=p<\frac{1}{2}$ and $\pi(0)=1-p(=q)$. Then (since we know ρ_2 for the Markov Chain on the weighted two point space) it is easy to see that $\rho_\infty = \dfrac{q-p}{pq(\log q-\log p)}$. Now, $\lambda_\infty = \dfrac{1}{pq}$, $\ell_\infty^+(\varepsilon)=\dfrac{1+\varepsilon}{-\log p}$, and $\ell_\infty^-(\varepsilon)=\dfrac{1-p+p\varepsilon}{-p\log p}$. Thus, as $p\to 0$, $\rho_\infty \sim -\dfrac{1}{p\log p}$, while the right hand side of (3.5) is of order $\dfrac{1}{-\sqrt{p}\log p}$ as $p\to 0$. So, in that case, (3.5) does not give the precise order, however, as shown next, the ε-version of (3.5), i.e., (3.25) does. For $p=+\infty$, the right hand side of (3.25) is

$$\sup_{\varepsilon\geq 0} \frac{(1+\varepsilon)}{2\sqrt{pq}\left(-\sqrt{2}\log p-\varepsilon\sqrt{pq}\log p+2(1+\varepsilon)\sqrt{pq}\right)},$$

and as $\varepsilon\to+\infty$, we get $\dfrac{1}{-2pq\log p+4pq}$, which is of the right order $\dfrac{1}{-p\log p}$ as $p\to 0$. Note also that (3.8) provides the right order $\dfrac{1}{-p\log p}$. Finally, for the Markov chain case where $\rho_2=\dfrac{q-p}{\log q-\log p}\sim\dfrac{1}{-\log p}$ as $p\to 0$, we have $\lambda_2=2$, $\ell_1^+=\dfrac{1-p}{-\log p}$ and thus (3.5) already provides the right order in p and so (3.25) gives little improvement.

(ii) Further lower bounds on (3.25) and (3.26) can be obtained using (3.21). For example for the version of (3.25) with $1\leq p<+\infty$, we get

$$\rho_p \geq \sup_{\varepsilon\geq 0} \frac{\lambda_p(\ell_1^+ + r\varepsilon)}{2\left(\sqrt{2\lambda_p}+\varepsilon+2\ell_1^+ + 2r\varepsilon\right)} \geq \frac{\lambda_p r}{2(1+2r)} = \frac{\lambda_p}{2(2-\log \pi^*)}, \tag{3.27}$$

where $r = \inf_{0<\pi(A)\le 1/2} \frac{1}{-\log \pi(A)}$, with $0<\pi^* = \inf_x \pi(x)$. As (i) above shows, (3.27) can be sharp and sometimes improves on (3.5) or (3.6). However (3.5) or (3.6) can also be better, indeed for the Cartesian product on n Markov chains, then (see Section 5) the right hand side of (3.5) (for $p=2$) is of order $n^{-3/2}$ while the right hand side of (3.27) is of order n^{-2}.

A bound better than (3.27) (but equivalent to it) is already known (at least for $p = 2$ and reversible Markov chains). Indeed, [7] obtained as a limiting case of hypercontractivity

$$\rho_2 \ge \frac{\lambda_2}{(2 - \log \pi^*)}. \tag{3.28}$$

(3.28) can also be obtained by our methods. Let

$$\ell_1(\varepsilon) = \inf_{0<\pi(A)\le \frac{1}{2}} \left\{ \frac{E|\nabla_1 \mathbf{1}_A|}{-\pi(A)\log \pi(A)} + \frac{\varepsilon}{-\log \pi(A)} \right\}, \tag{3.29}$$

then by Lemma 1 (and since the above infimum is also over $0<\pi(A)<1$),

$$\ell_1(\varepsilon) = \inf_{f\ge 0} \left\{ \frac{E|\nabla_1 f|}{\mathrm{Ent} f} + \varepsilon \frac{Ef}{\mathrm{Ent} f} \right\}.$$

Then, as in obtaining (3.13) gives

$$\rho_2 \ge \sup_{\varepsilon \ge 0} \frac{\lambda_2 \ell_1(\varepsilon)}{2\sqrt{\lambda_2} + \varepsilon + 2\ell_1(\varepsilon)},$$

and proceeding as in (3.27) will give (3.28). Results similar to (3.27) also hold for the other cases as well as for the ε-versions of (3.12)–(3.16) by using the corresponding version of (3.21). The still better (but still equivalent)

$$\rho_2 \ge \frac{\lambda_2(1 - 2\pi^*)}{\log(1/\pi^* - 1)},$$

obtained in [7] might follow from these and some tightening up of constants.

4. Gaussian bounds

The preceding arguments are based on the equivalence between some analytic and isoperimetric inequalities (Theorem 1) and essentially rely on the co-area (in)equality (Lemma 1). Still relying on Lemma 1, a purely isoperimetric bound can be obtained. Part of this approach is based on arguments introduced by Ledoux [14] in a differential geometric framework.

Let I_γ be the Gaussian isoperimetric function, i.e., $I_\gamma(x) = \varphi(\Phi^{-1}(x))$; where φ is the standard normal density and Φ^{-1} the inverse of its distribution function. Recall that I_γ is concave, defined on $[0,1]$ symmetric with respect to $1/2$ and with maximum $I_\gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2\pi}}$. Moreover, $\lim_{x\to 0^+} \frac{I_\gamma(x)}{x\sqrt{-2\log x}} = 1$ and, $I_\gamma(x) \geq 2\sqrt{\frac{2}{\pi}}x(1-x)$. Motivated by the previous definitions of ℓ, we now define the corresponding Gaussian isoperimetric constants (or Gaussian conductance) as:

$$g_p^+ = \inf_{0<\pi(A)\leq 1/2} \frac{E\nabla_p^+\mathbf{1}_A}{\pi(A)\sqrt{-\log\pi(A)}}; \tag{4.1}$$

with similar definitions for g_p^-, replacing ∇_p^+ by ∇_p^-; and

$$g_p = \inf_{0<\pi(A)\leq 1/2} \frac{E|\nabla_p\mathbf{1}_A|}{\pi(A)\sqrt{-\log\pi(A)}}, \tag{4.2}$$

$1\leq p\leq +\infty$ and again in (4.2) the infimum could be taken over $0<\pi(A)<1$. Note that $g\geq\ell$, and that for example $\ell_\infty^+ \geq (g_\infty^+)^2$.

Theorem 4. *For $p=1$ or $+\infty$,*

$$\rho_p \geq C(g_p^+)^2, \tag{4.3}$$

for some absolute constant C ($C=1/50$ will do).

Proof. As in the proof of Theorem 2, we prove the lower bound in (4.3) for $p=1$ and K Markovian, the case $p=+\infty$ being similar. Following Ledoux [14], from Lemma 1, for any function $k\geq 0$,

$$g_1^+ \int_{s_0}^{\infty} \pi(k\geq s)\sqrt{-\log\pi(k\geq s)}\,ds \leq E\nabla_1^+ k, \tag{4.4}$$

where s_0 is such that $\pi(k\geq s_0)\leq\frac{1}{2}$. Now, take $f\geq 0$ such that $Ef^2=1$ and apply (4.4) to $k=f^2\sqrt{\log(a+f^2)}$, where $a>1$ will be chosen later.

For this k, let us estimate the right hand side of (4.4). Since $x\sqrt{\log(a+x)}$ is increasing and convex for $x\geq 0$,

$$\begin{aligned} &E\nabla_1^+ k \\ &= \sum_x\sum_y \left(f^2(x)\sqrt{\log(a+f^2(x))} - f^2(y)\sqrt{\log(a+f^2(y))}\right)^+ K(x,y)\pi(x) \end{aligned}$$

$$\leq \sum_x \sum_y (f^2(x) - f^2(y))^+ \left(\sqrt{\log(a+f^2(x))} + \frac{f^2(x)}{2\sqrt{\log(a+f^2(x))}} \frac{1}{a+f^2(x)} \right)$$
$$K(x,y)\pi(x)$$
$$\leq \sum_x 2f(x) \left(\sqrt{\log(a+f^2(x))} + \frac{f^2(x)}{a+f^2(x)} \frac{1}{2\sqrt{\log(a+f^2(x))}} \right)$$
$$\sum_y (f(x) - f(y))^+ K(x,y)\pi(x))$$
$$= Ef(\nabla_1^+ f) \left(2\sqrt{\log(a+f^2)} + \frac{f^2}{a+f^2} \frac{1}{\sqrt{\log(a+f^2)}} \right). \tag{4.5}$$

Now for any $c > 0$;

$$Ef^2 \log(a+f^2) \leq \log(a+c^2) Ef^2 \mathbf{1}_{f<c} + E\mathbf{1}_{f\geq c} f^2 \log(a+f^2), \tag{4.6}$$

and we need to estimate the right most term in (4.6). To do that, let h be such that $h\left(z\sqrt{\log(a+z)}\right) = z\log(a+z)$. Then

$$\begin{aligned} h'\left(z\sqrt{\log(a+z)}\right) &= \sqrt{\log(a+z)} \frac{\log(a+z) + \frac{z}{a+z}}{\log(a+z) + \frac{z}{2(a+z)}} \\ &\leq \sqrt{\log(a+z)} \frac{\log(a+c^2) + \frac{c^2}{a+c^2}}{\log(a+c^2) + \frac{c^2}{2(a+c^2)}} \\ &= C_1(a,c)\sqrt{\log(a+z)}, \qquad \text{for } z \geq c^2. \end{aligned}$$

Thus,

$$E\mathbf{1}_{f\geq c} f^2 \log(a+f^2) = E\mathbf{1}_{f\geq c} h(k)$$
$$= c^2 \log(a+c^2) E\mathbf{1}_{f\geq c} + \int_{c^2\sqrt{\log(a+c^2)}}^{\infty} h'(s)\pi(k\geq s)ds. \tag{4.7}$$

But, if $s = x^2\sqrt{\log(a+x^2)}$, then since $Ef^2 = 1$, $\pi(k \geq s) = \pi(f \geq x) \leq \frac{1}{x^2}$, it follows that $-\log \pi(k\geq s) \geq 2\log x$. Moreover, for all $x \geq c > 1$,

$$\begin{aligned} h'(s) &\leq C_1(a,c)\sqrt{\log(a+x^2)} \\ &\leq C_1(a,c)\sqrt{\frac{\log(a+x^2)}{\log x^2}}\sqrt{-\log \pi(k\geq s)} \\ &\leq C_1(a,c)\sqrt{\frac{\log(a+c^2)}{\log c^2}}\sqrt{-\log \pi(k\geq s)}, \end{aligned} \tag{4.8}$$

for $x \geq c$. Using (4.8) into (4.7), and for $\frac{1}{c^2} \leq \frac{1}{2}$, i.e., for $c \geq \sqrt{2}$, we have

$$\begin{aligned} &E\mathbf{1}_{f\geq c} f^2 \log(a+f^2) \\ &\quad \leq c^2 \log(a+c^2)\pi(f \geq c) \\ &\quad +C_1(a,c)\sqrt{\frac{\log(a+c^2)}{\log c^2}} \int_{s_0}^{\infty} \pi(k \geq s)\sqrt{-\log \pi(k \geq s)}ds \end{aligned}$$

$$\leq c^2 \log(a+c^2)\pi(f \geq c) + \frac{C_1(a,c)}{g_1^+}\sqrt{\frac{\log(a+c^2)}{\log c^2}} E\nabla_1^+ k. \tag{4.9}$$

Combining (4.6) and (4.9) gives (since $Ef^2 = 1$)

$$\begin{aligned} &Ef^2 \log(a+f^2) \\ &\leq \log(a+c^2)(Ef^2\mathbf{1}_{f<c} + c^2\pi(f \geq c)) + \frac{C_1(a,c)}{g_1^+}\sqrt{\frac{\log(a+c^2)}{\log c^2}} E\nabla_1^+ k \\ &\leq \log(a+c^2) + \frac{C_2(a,c)}{g_1^+} Ef(\nabla_1^+ f)\sqrt{\log(a+f^2)}\left(2 + \frac{f^2}{a+f^2}\frac{1}{\log(a+f^2)}\right) \\ &\leq \log(a+c^2) + \frac{C_3(a,c)}{g_1^+} Ef(\nabla_1^+ f)\sqrt{\log(a+f^2)}, \end{aligned}$$

by (4.5), and where

$$C_3(a,c) = C_1(a,c)\sqrt{\frac{\log(a+c^2)}{\log c^2}}\left(2 + \max_{x\geq 0}\frac{x}{a+x}\frac{1}{\log(a+x)}\right).$$

Hence,

$$Ef^2 \log(a+f^2) \leq \log(a+c^2) + \frac{C_3(a,c)}{g_1^+}\sqrt{E(\nabla_1^+ f)^2}\sqrt{Ef^2\log(a+f^2)}. \tag{4.10}$$

Completing the square this leads to

$$Ef^2 \log(a+f^2) \leq 2\log(a+c^2) + \frac{C_4(a,c)}{g_1^{+2}} E(\nabla_1^+ f)^2, \tag{4.11}$$

where

$$\begin{aligned} &C_4(a,c) \\ &\quad = \frac{\log(a+c^2)}{\log c^2}\left(2+\max_{x\geq 0}\frac{x}{a+x}\frac{1}{\log(a+x)}\right)^2\left(\frac{2(a+c^2)\log(a+c^2)+2c^2}{2(a+c^2)\log(a+c^2)+c^2}\right)^2. \end{aligned}$$

Choosing $a=3$ and $2\log(a+c^2)=5$, we see that $C_4(a,c)<8$. Thus, for any $f\geq 0$, $Ef^2=1$

$$Ef^2\log(3+f^2)\leq 5+\frac{8}{g_1^{+2}}E(\nabla_1^+ f)^2.$$

Hence, by homogeneity,

$$\text{Ent } f^2 \leq 5Ef^2+\frac{8}{g_1^{+2}}E(\nabla_1^+ f)^2, \tag{4.12}$$

for any $f\geq 0$. Now taking $f : X \longrightarrow \mathbf{R}$, and applying (4.12) to f^+ and f^-, using $\text{Ent } f^2 \leq \text{Ent } f^{+2}+\text{Ent } f^{-2}$ and $\nabla_1^+ f^+ \leq |\nabla_1 f|\mathbf{1}_{f>0}$, $\nabla_1^+ f^- \leq |\nabla_1 f|\mathbf{1}_{f<0}$, give

$$\text{Ent } f^2 \leq 5Ef^2+\frac{8}{g_1^{+2}}E|\nabla_1 f|^2,$$

for all f. Again Rothaus' inequality, [17] (Lemma 9), leads to

$$\begin{aligned}\text{Ent } f^2 &\leq \text{Ent } (f-Ef)^2+2 \text{ Var } f\\ &\leq 7 \text{ Var } f+\frac{8}{g_1^{+2}}E|\nabla_1 f|^2\\ &\leq \left(\frac{7}{\lambda_1}+\frac{8}{g_1^{+2}}\right)E|\nabla_1 f|^2, \end{aligned}\tag{4.13}$$

and

$$\rho_1 \geq \frac{\lambda_1 g_1^{+2}}{7g_1^{+2}+8\lambda_1}. \tag{4.14}$$

Next, we use a "non–standard" form of Cheeger's inequality, namely, $\lambda_1\geq \frac{(h_1^+)^2}{4}$, where $\lambda_1=\inf_{f\neq constant}\frac{E|\nabla_1 f|^2}{\text{Var} f}$ and where $h_1^+=\inf_{0<\pi(A)\leq 1/2}\frac{E\nabla_1^+\mathbf{1}_A}{\pi(A)}$ (see [10] for a proof of a general inequality of this type). Actually to prove the above, it is enough to use Theorem 1 and the usual ingredients present in the proof of the "standard" Cheeger's inequality.

Finally, $\frac{(h_1^+)^2}{4}\geq\frac{\log 2}{4}g_1^{+2}$, and so

$$\rho_1 \geq \frac{\log 2}{28+2\log 2}g_1^{+2}. \tag{4.15}$$

This proves (4.3) for $p=1$. In the case $p=+\infty$, one proceeds as above where in the next to last step one uses (with obvious notation) $\lambda_\infty\geq\frac{(h_\infty^+)^2}{4}$, which is proved in [5]. ∎

Remark 5. (i) The proof of (4.3) actually shows that

$$\rho_p \geq C\, \frac{\lambda_p \min(g_1^{+2}, g_\infty^{+2})}{\lambda_p + \min(g_1^{+2}, g_\infty^{+2})}, \quad 1 \leq p < +\infty, \tag{4.16}$$

and that

$$\rho_\infty \geq C\, \frac{\lambda_\infty g_\infty^{+2}}{\lambda_\infty + g_\infty^{+2}}, \tag{4.17}$$

for some absolute constant C. Further lower bounds follow by replacing in (4.17) λ_∞ by $\dfrac{(h_\infty^+)^2}{4}$ and in (4.16), and for $p=1$, λ_1 by $\dfrac{(h_1^+)^2}{4}$.

In view of the results in [5], it is natural to believe that the "minus" version of (4.3) is

$$\rho_\infty \geq C \left(\sqrt{g_\infty^- + 1} - 1\right)^2. \tag{4.18}$$

However, a lower bound as above, i.e., of order $\min(g_\infty^-, g_\infty^{-2})$, does not generally hold. Again, on the weighted two point space with $\pi(1) = p < 1/2$, $\pi(0) = 1 - p (= q)$, we have $\lambda_\infty = \dfrac{1}{pq}$, $\rho_\infty = \dfrac{p-q}{pq(\log p - \log q)}$, $\ell_\infty^+ = \dfrac{1}{-\log p}$, $g_\infty^+ = \dfrac{1}{\sqrt{-\log p}}$, and $g_\infty^- = \dfrac{q}{p\sqrt{-\log p}}$. This shows that (4.18) cannot hold.

Nevertheless, one might observe that from Theorem 2 and the results in [5], it follows that

$$\rho_\infty \geq C \min(\ell_\infty^-, \ell_\infty^{-2}),$$

for some absolute constant C. Again, the weighted two point space shows that this last inequality is tight since then $\ell_\infty^- = \dfrac{q}{-p\log p}$.

(ii) Coming back to the two examples of the previous section, it is easily seen that for the simple Markov chain on the complete graph, g_1^+ is of order $\dfrac{1}{\sqrt{\log n}}$ and so (4.3), for $p=1$, is sharp in n. Moreover, for the simple random walk on the discrete cube, g_∞^+ is of order $\dfrac{1}{\sqrt{n}}$ and so (4.3), for $p = +\infty$, is sharp in n.

(iii) Again trivial upper bounds are obtained by applying the defining property of the log-Sobolev constants to $f = \mathbf{1}_A$ and using the generic facts $\ell \leq g$ and $\rho \leq \dfrac{\lambda}{2}$.

(iv) An ε–version of Theorem 4 is also true. This follows from a long and tedious computation. Let us briefly indicate how to get these estimates (in the Markovian case). First,

$$g_1^+(\varepsilon) = \inf_{0<\pi(A)\le\frac{1}{2}} \left\{ \frac{E\nabla_1^+\mathbf{1}_A}{\pi(A)\sqrt{-\log\pi(A)}} + \frac{\varepsilon}{\sqrt{-\log\pi(A)}} \right\}, \tag{4.19}$$

and similarly for $p=+\infty$. With this definition start the proof as in (4.4) with an extra εEk on the right hand side. Carrying this extra term, (4.10) becomes

$$\begin{aligned} Ef^2\log(a+f^2) &\le \log(a+c^2) + \frac{C_3(a,c)\sqrt{E(\nabla_1^+ f)^2 Ef^2\log(a+f^2)}}{g_1^+(\varepsilon)} \\ &\quad + \frac{C_2(a,c)\varepsilon\sqrt{Ef^2\log(a+f^2)}}{g_1^+(\varepsilon)}. \end{aligned} \tag{4.20}$$

Completing the square and simple inequalities lead to

$$\begin{aligned} &Ef^2\log(a+f^2) \\ &\le 2\log(a+c^2) + \frac{2}{g_1^{+2}(\varepsilon)}\left(C_4(a,c)E(\nabla_1^+ f)^2 + C_2^2(a,c)\varepsilon^2\right), \end{aligned} \tag{4.21}$$

where $C_2(a,c)$ and $C_4(a,c)$ are the same as previously defined. Taking the same values for a and c as before, and since $4C_2^2(a,c) < C_4(a,c) < 8$ we see that (4.12) becomes

$$\operatorname{Ent} f^2 \le \left(5 + \frac{4\varepsilon^2}{g_1^{+2}(\varepsilon)}\right) Ef^2 + \frac{16}{g_1^{+2}(\varepsilon)} E(\nabla_1^+ f)^2, \tag{4.22}$$

for any $f\ge 0$. From this it follows that

$$\rho_p \ge \sup_{\varepsilon\ge 0} \frac{\lambda_p g_1^{+2}(\varepsilon)}{16\lambda_p + 7g_1^{+2}(\varepsilon) + 4\varepsilon^2}, \quad 1\le p<+\infty. \tag{4.23}$$

which is the ε–version of (4.16), for Markov kernels. Note also that further lower bounds on (4.23) can be obtained by lowerbounding λ_p in terms of $g_1^{+2}(\varepsilon)$. The other cases (p, minus) can be proved in a similar way.

As a final note, it is easy to find examples where (4.23) improves (4.16) or (4.3). On the weighted 2–point spaces, and as previously indicated, ρ_∞ is of order $\dfrac{1}{-p\log p}$, which is the order of the above right hand sides, while the right hand sides of (4.3) or (4.17) only give $\dfrac{1}{-\log p}$.

5. Concluding remarks

• It is not difficult to see ([11]) that $\lambda_p(K^n)$ and $\rho_p(K^n)$, the spectral gap and the log-Sobolev constant of the Cartesian product (X^n, K^n, π^n) of the Markov chain (X, K, π) are such that

$$\lambda_p(K^n) = \frac{\lambda_p(K)}{n^{2/p}}; \qquad \rho_p(K^n) = \frac{\rho_p(K)}{n^{2/p}}; \qquad 1 \le p \le 2$$

and

$$\lambda_p(K^n) \ge \frac{\lambda_p(K)}{n}; \qquad \rho_p(K^n) \ge \frac{\rho_p(K)}{n}; \qquad 2 < p \le +\infty,$$

and similarly for $\lambda_p^{\pm}(K^n)$ and $\rho_p^{\pm}(K^n)$. In view of these, of the estimates of Theorems 2 and 4; and by "homogeneity," it is rather natural to guess that $\ell_p(K^n) \approx \frac{\ell_p(K)}{n^{1/p}}$; $1 \le p \le 2$ and $\ell_p(K^n) \approx \frac{\ell_p(K)}{\sqrt{n}}$; $2 < p \le +\infty$ and similarly for $\ell_p^{\pm}$, g_p and $g_p^{\pm}$. For $p = 1$, this is indeed true; let us prove it for ℓ_1. First, it is clear (taking $f = \mathbf{1}_A, A \subset X$) that $\ell_1(K^n) \le \frac{\ell_1(K)}{n}$. Next, for the converse inequality, $\ell_1(K^n) \ge \frac{\ell_1(K)}{n}$, note that ([11]) for $f : (X^n, K^n, \pi^n) \longrightarrow \mathbf{R}$ and $1 \le p < +\infty$,

$$n^{2/p}(D_p f)^2 = \left(\sum_{i=1}^{n} (D_p^{(i)} f)^p\right)^{2/p} \ge \min\left(1, n^{\frac{2}{p}-1}\right) \sum_{i=1}^{n} (D_p^{(i)} f)^2, \tag{5.1}$$

while for $p = +\infty$,

$$\max_{1 \le i \le n} (D_\infty^{(i)} f) = (D_\infty f), \tag{5.2}$$

where for any $f : X^n \longrightarrow \mathbf{R}$, $D_p^{(i)} f$ is the one dimensional gradient obtained by fixing all but the ith coordinate of f and where D is any one $\nabla^{\pm}$ or $|\nabla|$. Then, to obtain the result, use the well known tensorization property of the entropy: $\mathrm{Ent} f \le E_{\pi^n} \sum_{i=1}^n \mathrm{Ent}_{\pi_i} f$ (see [15], [16], [10], [11], ...). For the other values of p, we have not been able to assert how true our "guesses" are.

• As noted by Bobkov (see [15]), a functional $\mathcal{L}$ of the form $\mathcal{L}f = E_{\pi^n}\mathcal{R}(f) - \mathcal{R}(E_{\pi^n} f)$, tensorizes if and only if it is convex. Thus, for such functionals we also have

$$\inf_{A \subset X^n} \frac{E_{\pi^n} |\nabla_1 \mathbf{1}_A|}{\mathcal{L}\mathbf{1}_A} = \frac{1}{n} \inf_{A \subset X} \frac{E_\pi |\nabla_1 \mathbf{1}_A|}{\mathcal{L}\mathbf{1}_A}. \tag{5.3}$$

The convexity of the functional $\mathcal{L}$ implies that the function $\mathcal{R}$ is convex. Thus, if $\mathcal{R}'' > 0$, $\mathcal{L}$ tensorizes if and only if $1/\mathcal{R}''$ is concave. Examples of such $\mathcal{L}$ are the variance ($\mathcal{R}(x) = x(1+x)$), the entropy ($\mathcal{R}(x) = x \log x$), etc...

For Cartesian products of graphs, (5.3) is also true without the factor $1/n$; related results of this type also appear in [20].

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